ECS 455: Problem Set 3 Solution

1. Complete the following M/M/m/m description with the following terms:

   (I) Bernoulli (II) binomial (III) exponential
   (IV) Gaussian (V) geometric (VI) Poisson

   The Erlang B formula is derived under some assumptions. Two important assumptions are (1) the call request process is modeled by a/an **Poisson** process and (2) the call durations are assumed to be i.i.d. **exponential** random variables. For the call request process, the times between adjacent call requests can be shown to be i.i.d. **exponential** random variables. On the other hand, if we consider non-overlapping time intervals, the numbers of call requests in these intervals are **Poisson** random variables.

   In order to analyze or simulate the system described above, we consider slotted time where the duration of each time slot is small. This technique shifts our focus from continuous-time Markov chain to discrete-time Markov chain. In the limit, for the call request process, only one of the two events can happen during any particular slot: either (1) there is one new call request or (2) there is no new call request. When the slots are small and have equal length, the numbers of new call requests in the slots can be approximated by i.i.d. **Bernoulli** random variables. In which case, if we count the total number of call requests during n slots, we will get a/an **binomial** random variable because it is a sum of i.i.d. **Bernoulli** random variables.

   When we consider a particular time interval I (not necessarily small), the number of slots in this interval will increase as the slots get smaller. In the limit, the number of call requests in the time interval I which we approximated by a **binomial** random variable before will approach a/an **Poisson** random variable.

   Similarly, if we consider the numbers of slots between adjacent call requests, these number will be i.i.d. **geometric** random variables. These random variables can be thought of as discrete counterparts of the i.i.d. **exponential** random variables in the continuous-time model.

   Some term(s) above is/are used more than once. Some term(s) is/are not used.

HW3 Q2

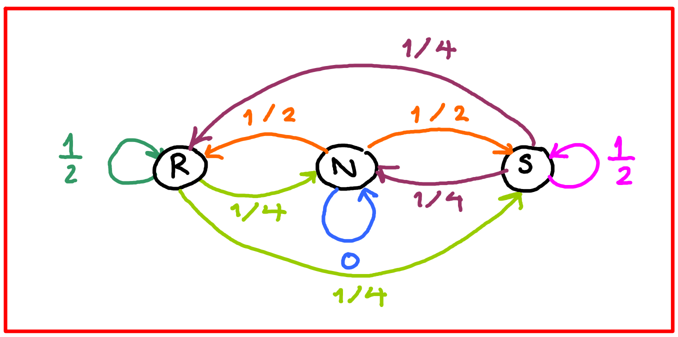
Tuesday, January 11, 2011 8:52 AM

(a)

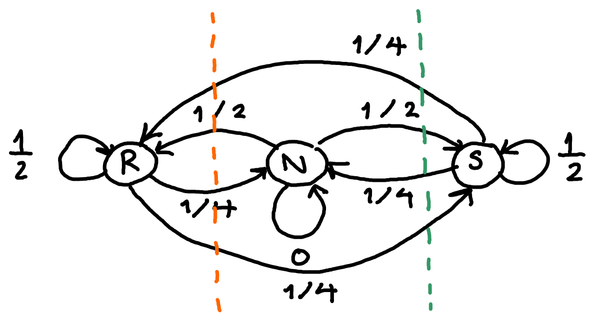
As hinted, we draw three states.

Next, follow the description sentence by sentence to get the transition probabilities.

1. Never have two nice days in a row
2. If have a nice day, just as likely to have snow as rain the next day
3. If have snow or rain, they have an even chance of having the same the next day.
4. If there is change from snow or rain, only half of the time is this a change to a nice day.



(b)



$$\frac{1}{2} p_N + \frac{1}{4} p_S = \frac{1}{4} p_R + \frac{1}{4} p_R$$

$$\frac{1}{2} p_N + \frac{1}{4} p_S = \frac{1}{2} p_R$$

$$-\frac{1}{2} p_R + \frac{1}{2} p_N + \frac{1}{4} p_S = 0$$

$$\frac{1}{4} p_S + \frac{1}{4} p_S = \frac{1}{2} p_N + \frac{1}{4} p_R$$

$$\frac{1}{2} p_S = \frac{1}{2} p_N + \frac{1}{4} p_R$$

$$\frac{1}{4} p_R + \frac{1}{2} p_N - \frac{1}{2} p_S = 0$$

One more equation: $p_R + p_N + p_S = 1$

Solve 3 eqns, 3 unknowns.

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Solve 3 eqns, 3 unknowns.

$p_R = 0.4$, $p_N = 0.2$, and $p_S = 0.4$

(c) "365 days" is a long time.

The probability of being a nice day $= p_N = 0.2$

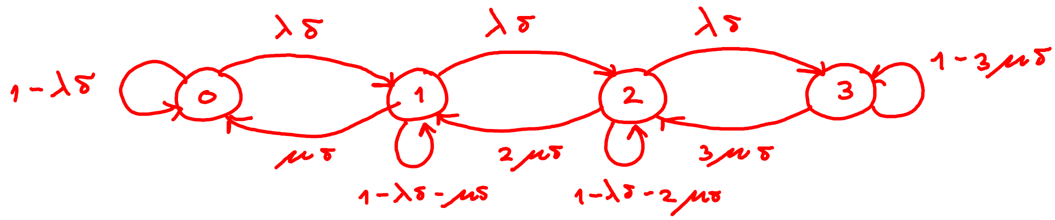
HW3 Q3

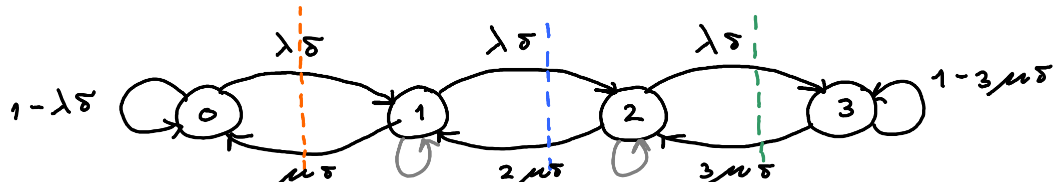
Tuesday, February 14, 2012 1:04 PM

$m = 3$

(a) Erlang B model

Markov chain:





$p_0 \lambda \cancel{\delta} = p_1 \mu \cancel{\delta}$ $\qquad$ $p_1 \lambda \cancel{\delta} = p_2\, 2\mu \cancel{\delta}$ $\qquad$ $p_2 \lambda \cancel{\delta} = 3\mu \cancel{\delta}\, p_3$

$p_1 = A p_0$ $\qquad$ $p_2 = \frac{A}{2} p_1 = \frac{A^2}{2} p_0$ $\qquad$ $p_3 = \frac{A}{3} p_2 = \frac{A^3}{3!} p_0$

$$p_0 + p_1 + p_2 + p_3 = 1 \quad \Rightarrow \quad p_0 = \left(1 + A + \frac{A^2}{2} + \frac{A^3}{3}\right)^{-1}$$

$$A = \frac{\lambda}{\mu} = \lambda \times \frac{1}{\mu} = \left(\overset{2}{\cancel{10}}\ \frac{\text{calls}}{\text{hour}} \times \frac{1\ \text{hour}}{\underset{\cancel{5}}{\cancel{60}}\ \text{mins}}\right) \times \left(\cancel{12}\ \text{mins}\right) = 2 \ \text{Erlangs}.$$

$$\Rightarrow p_0 = \frac{3}{19},\ p_1 = \frac{6}{19},\ p_2 = \frac{6}{19},\ \underbrace{p_3 = \frac{4}{19}}$$

$$\Downarrow$$

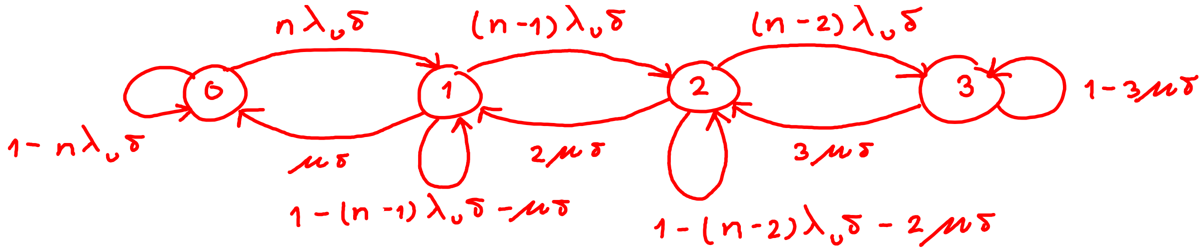
$$\text{call blocking probability} = \frac{4}{19} \approx 0.211$$

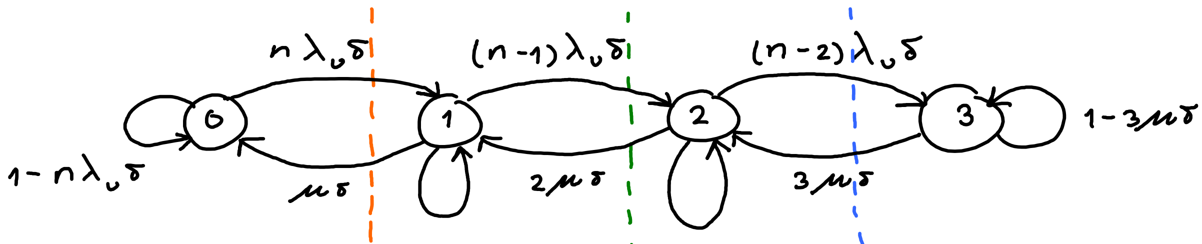
(b) and (c)

Observe that $\lambda_u \times n = \lambda$ in part (a).

So, $\lambda_u = \frac{\lambda}{n}$ and $A_u = \frac{A}{n} = \frac{2}{n}$ Erlangs.

Markov chain:





$p_0 n\lambda_u\cancel{\delta} = p_1\mu\cancel{\delta}$ $\qquad$ $p_1(n-1)\lambda_u\cancel{\delta} = 2\mu\cancel{\delta}\, p_2$ $\qquad$ $p_2(n-2)\lambda_u\cancel{\delta} = p_3\, 3\mu\cancel{\delta}$

$p_1 = nA_u$

$$p_2 = \frac{(n-1)}{2}A_u p_1 = \frac{n(n-1)}{2}A_u^2\, p_0$$

$$p_3 = \frac{(n-2)}{3}A_u p_2 = \frac{n(n-1)(n-2)A_u^3}{3!}p_0$$

$$p_0+p_1+p_2+p_3 = 1 \Rightarrow \begin{cases} p_0 = \frac{25}{131},\ p_1 = \frac{50}{131},\ p_2 = \frac{40}{131},\ p_3 = \frac{16}{131} & \text{when } n=5, \\ p_0 = 0.159,\ p_1 = 0.319,\ p_2 = 0.316,\ p_3 = 0.206 & \text{when } n=100. \end{cases}$$

As discussed in class, the call blocking probability is given by

$$\frac{(n-m)p_m}{\sum_{k=0}^{m}(n-k)p_k} = \frac{(n-3)p_3}{np_0+(n-1)p_1+(n-2)p_2+(n-3)p_3}$$

$$= \begin{cases} \frac{32}{477} \approx 0.067 & \text{when } n=5 \\ 0.203 & \text{when } n=100 \end{cases}$$

close to the answer from Erlang B.

Remark:

For those who are interested in why the Engset model converges to the Erlang B model when $n\to\infty$, read on.

Note that

$$p_k = \binom{n}{k} A_u^k p_o = \binom{n}{k} \frac{A^k}{n^k} p_o = \frac{n!}{(n-k)!\,k!\,n^k} A^k p_o .$$

For fixed k,

$$\frac{n!}{(n-k)!\,n^k} = \frac{n \times (n-1) \times \cdots \times (n-(k-1))}{n^k} = \frac{n}{n} \times \frac{n-1}{n} \times \cdots \times \frac{n-(k-1)}{n}$$

$$\longrightarrow 1 \quad \text{as} \quad n \to \infty .$$

Hence,

$$p_k = \frac{\binom{n}{k} \frac{A^k}{n^k}}{\sum_{i=0}^{m} \binom{n}{i} \frac{A^i}{n^i}} \longrightarrow \underbrace{\frac{\frac{1}{k!} A^k}{\sum_{i=0}^{m} \frac{1}{i!} A^i}}_{\text{same as the steady-state probabilities in Erlang B model.}} \quad \text{as} \quad n \to \infty$$

Similarly, for the call blocking probability,

$$P_{CB} = \frac{(n-m)\, p_m}{\sum_{k=0}^{m} (n-k)\, p_k} = \frac{(n-m) \binom{n}{m} \frac{A^m}{n^m} \cancel{p_o}}{\sum_{k=0}^{m} (n-k) \binom{n}{k} \frac{A^k}{n^k} \cancel{p_o}} = \frac{\binom{n}{m} \frac{A^m}{n^m}}{\sum_{k=0}^{m} \frac{n-k}{n-m} \binom{n}{k} \frac{A^k}{n^k}}$$

$$\longrightarrow \underbrace{\frac{\frac{A^m}{m!}}{\sum_{k=0}^{m} \frac{A^k}{k!}}}_{} \quad \text{as} \quad n \to \infty$$

same as the call blocking probability in Erlang B model.

HW3 Q4

Wednesday, January 05, 2011 3:21 PM

(a)

$$P_m = \frac{\binom{n}{m} A_u^m}{\sum_{k=0}^{m} \binom{n}{k} A_u^k} = \frac{\sum_{k=0}^{m} \binom{n}{k} A_u^k - \sum_{k=0}^{m-1} \binom{n}{k} A_u^k}{\sum_{k=0}^{m} \binom{n}{k} A_u^k}$$

$$= \frac{z(m,n) - z(m-1,n)}{z(m,n)} = 1 - \frac{z(m-1,n)}{z(m,n)}$$

Hence, $c = 1$

(b)

First, note that

$$(n-k) \times \binom{n}{k} = (n-k) \times \frac{n!}{k!(n-k)!} = \frac{n!}{k!(n-k-1)!}$$

$$= \frac{n \times (n-1)!}{k!\,(n-1-k)!} = n \binom{n-1}{k}$$

Therefore,

$$P_b = \frac{(n-m)\binom{n}{m} A_u^m}{\sum_{k=0}^{m} (n-k)\binom{n}{k} A_u^k} = \frac{\cancel{n}\binom{n-1}{m} A_u^m}{\sum_{k=0}^{m} \cancel{n}\binom{n-1}{k} A_u^k}$$

$$= \frac{\sum_{k=0}^{m} \binom{n-1}{k} A_u^k - \sum_{k=0}^{m-1} \binom{n-1}{k} A_u^k}{\sum_{k=0}^{m} \binom{n-1}{k} A_u^k}$$

$$= \frac{z(m,n-1) - z(m-1,n-1)}{z(m,n-1)} = 1 - \frac{z(m-1,n-1)}{z(m,n-1)}$$

Hence, $c_1 = c_2 = c_4 = 1$ and $c_3 = 0$.

(c)

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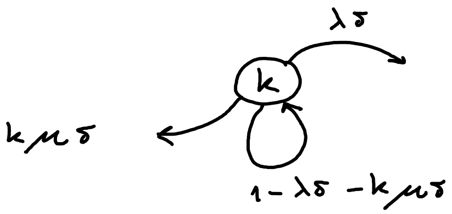
$$P_b = \frac{\binom{n-1}{m} A_u^m}{\sum_{k=0}^{m} \binom{n-1}{k} A_u^k} \overset{m = n-1}{=} \frac{\binom{m}{m} A_u^m}{\sum_{k=0}^{m} \binom{m}{k} A_u^k} = \frac{A_u^m}{(1+A_u)^m} = \left(\frac{A_u}{1+A_u}\right)^m$$

(a) Nothing changes from the M/M/m/m model when $k < m$.

We have

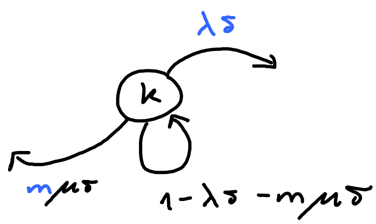


When $k \geq m$, the call request rate is still λ. The difference is that now we have a queue for the new requests to wait. (In M/M/m/m, these requests are discarded and the calls are blocked.)

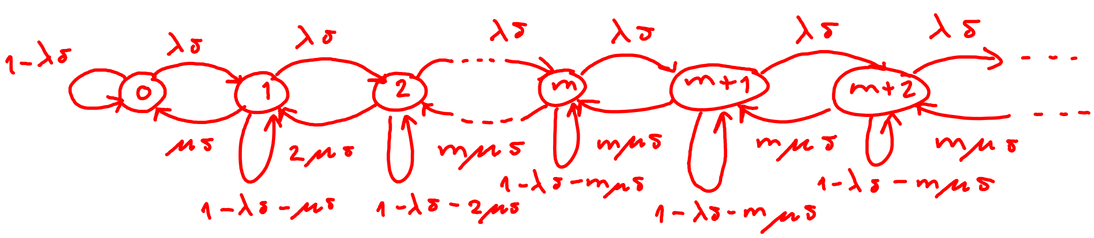
When $k \geq m$, all m channels are being used. There are $k-m$ requests waiting in the queue. When there is one new call request, it will be added to the queue and hence the system move from state k to $k+1$. Again, this new call request occurs with probability $\lambda\delta$ (approximately).

When $k \geq m$, all m channels are being used. There are m customers talking on the phone. So, the probability of one call ends is (approximately) $m\mu\delta$.

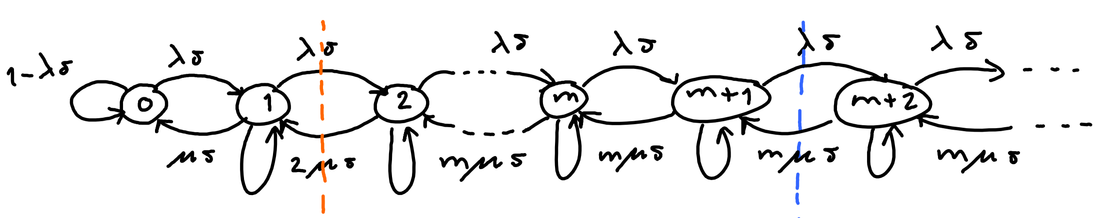
Therefore, when $k \geq m$, we have

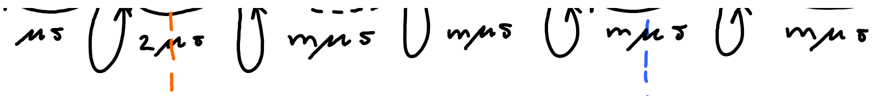


Markov chain :



(b)





$$p_{k-1}\,\lambda\cancel{s} = p_k\, k\mu\cancel{s}$$

$$p_k = \frac{A}{k} p_{k-1} = \frac{A^k}{k!} p_0 \quad \text{for } 0<k<m$$

$$p_{k-1}\,\lambda\cancel{s} = m\mu\cancel{s}\, p_k$$

$$p_k = \frac{A}{m} p_{k-1} \quad \text{for } k \geq m$$

$$\Rightarrow p_m = \frac{A}{m} p_{m-1} = \frac{A}{m}\frac{A^{m-1}}{(m-1)!} p_0 = \frac{A^m}{m!} p_0$$

$$p_k = \left(\frac{A}{m}\right)^{k-m} \frac{A^m}{m!} p_0 = \frac{A^k}{m!\,(m^{k-m})} p_0 = \frac{m^m}{m!}\left(\frac{A}{m}\right)^k p_0 \quad \text{for } k \geq m.$$

$$\sum_{k=0}^{m} p_k = 1 \Rightarrow 1 = \sum_{k=0}^{m-1} \frac{A^k}{k!} p_0 + \underbrace{\sum_{k=m}^{\infty} \frac{m^m}{m!}\left(\frac{A}{m}\right)^k p_0} = \sum_{k=0}^{m-1} \frac{A^k}{k!} p_0 + \frac{A^m}{m!\,(1-\frac{A}{m})} p_0$$

geometric series

$$= \begin{cases} \dfrac{m^m}{m!} \dfrac{\left(\frac{A}{m}\right)^m}{1-\frac{A}{m}} p_0 & \text{if } A < m \\ \infty & \text{if } A \geq m \end{cases}$$

$$\Rightarrow p_0 = \left(\left(\sum_{k=0}^{m-1} \frac{A^k}{k!}\right) + \frac{A^m}{m!\,(1-\frac{A}{m})} \right)^{-1}$$

Therefore,

$$p_k = \begin{cases} \dfrac{A^k}{k!} p_0, & k<m \\ \dfrac{A^k}{m!\,(m^{k-m})} p_0, & k \geq m \end{cases}$$

(c) Delayed call probability

$$= \sum^{\infty} p_k = \frac{A^m}{\ldots} \ldots p_0 = \frac{\dfrac{A^m}{m!\,(1-A/m)}}{\ldots}$$

$$= \sum_{k=m}^{\infty} p_k = \frac{A^m}{m!\left(1-\frac{A}{m}\right)} p_0 = \frac{\overline{m!\,(1-A/m)}}{\frac{A^m}{m!\left(1-\frac{A}{m}\right)} + \sum_{k=0}^{m-1} \frac{A^k}{k!}}$$

$$= \frac{A^m}{A^m + m!\left(1-\frac{A}{m}\right) \sum_{k=0}^{m-1} \frac{A^k}{k!}}$$

Remark: This formula is call the "Erlang C formula".